Talk 4 - Ringed Spaces

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4.1 Ringed Spaces

Definition 1. Let R be a ring. A *ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is the sheaf of R-algebras on X.

Example 2. The following pairs are ringed spaces:

(a) $(X, \mathbb{Z}_X)$ where $\mathbb{Z}_X$ is the constant sheaf on X. (for $R = \mathbb{Z}$)

(b) $(X, \mathscr{C}^{\mathbb{R}})$ where X is an arbitrary topological space and $\mathscr{C}^{\mathbb{R}}$ is the sheaf of continious ($\mathbb{R}$-valued) functions on X. (for $R = \mathbb{R}$)

(c) $(X, \mathscr{C}^r)$ where X is an arbitrary banach space and $\mathscr{C}^r$ is the sheaf of r-times continious differentiable functions (with values in $\mathbb{R}$). (for $R = \mathbb{R}$)

Definition 3. Let R be a ring, then a map $\Phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is *morphism of ringed spaces over R* given by a continious map $\phi : X \to Y$ together with a ϕ-morphism of sheaves of R-algebras.

Proposition 4. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces. The map $\Phi : (X, \mathcal{O}_x) \to (Y, \mathcal{O}_Y)$ is an *isomorphism of ringed spaces (over R)*

$\Leftrightarrow$ Φ is invertible

$\Leftrightarrow$ The underlying cont. map $\phi : X \to Y$ is an homeomorphism and $\mathcal{O}_Y \to \phi_* \mathcal{O}_X$ is an isomorphism of sheaves of R-algebras.

4.2 Geometric Spaces and Manifolds

Definition 5. Let R be a ring. A ringed space $(X, \mathcal{O}_X)$ (over R) is called a *geometric space* if and only if for all $x \in X$ the stalks $\mathcal{O}_{X,x}$ are local rings and the stalk maps $\Psi_x : \mathcal{O}_{Y,\phi(x)} \to \mathcal{O}_{X,x}$ are local morphisms of local rings. (i.e. $\Psi_x(m_{\phi(x)} \subset m_x)$

Remark 6. Morphisms of geometric space are just morphisms of ringed spaces. Therefor we get a sub categorie $GeoSp \subset RSP$ of ringed spaces.

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Example 7. The following pairs are geometric spaces:

(a) $(Spec(R), \mathcal{O}_{Spec(R)})$ where R is a ring. Thus all affine schemes are geomtric spaces.

(b) $(X, \mathscr{C}^F)$ where R is a ring, F an R-algebra (which is a topological field), X a topological space and $\mathscr{C}^F$ the sheaf of F-valued functions on X.

(c) $(X, \mathscr{C}^r)$ where $R = \mathbb{R}$, X a banach space and $\mathscr{C}^r$ the sheaf of r-times differentiable, $\mathbb{R}$-valued functions on X.

Definition 8. Fix a ring R and suppose we have geometric space $(M, \mathcal{O}_M)$ over R (to be regarded as a 'model'). We say that a geometric space $(X, \mathcal{O}_X)$ is *locally isomorphic* to $(M, \mathcal{O}_M)$ if and only if for all $x \in X$ exists an open neighbourhood $U \subset X$ of x and an open subset $V \subset M$ such that there exists an isomorphism $\varphi : (U, \mathcal{O}_{\mathcal{X}}|_U) \to (V, \mathcal{O}_M|_V)$.
Given $\mathcal{M}$ a class of model spaces $(M, \mathcal{O}_M)$ we say that $(X, \mathcal{O}_X)$ is a *manifold of type* $\mathcal{M}$ (over R) if and only if there exists an open cover $(U_i)_{i \in I}$ of X such that for all i the restriction of the sheaf $\mathcal{O}_X|_{U_i}$ is isomorphic to a model space $(M, \mathcal{O}_M)$ of $\mathcal{M}$.

Example 9. If the geometric space $(X, \mathcal{O}_X)$ (over R) is a manifold of type:

(a) $\mathcal{M} = \{(\mathbb{R}^n, \mathscr{C}^{\mathbb{R})})|n \in \mathbb{N}\}$ we obtain topological manifolds of dimension n. (for $R = \mathbb{R}$)

(b) $\mathcal{M} = \{(\mathbb{R}^n, \mathscr{C}^r)|n \in \mathbb{N}\}$ we obtain differential manifolds of class $r \in \mathbb{N}$ and dimension n. (for $R = \mathbb{R}$)

(c) $\mathcal{M} = \{(\mathbb{C}, \mathscr{C}^w)\}$ we obtain Riemann surfaces. (for $R = \mathbb{C}$)

(d) $\mathcal{M} = \{(\mathbb{C}^n, \mathscr{C}^w)|n \in \mathbb{N}\}$ we obtain complex analytic manifolds. (for $R = \mathbb{C}$)

(e) $\mathcal{M} = \{$*affine schemes*$\}$ we obtain schemes. (for $R = \mathbb{Z}$)